PERGAMON

Applied Mathematics Letters 13 (2000) 23–28

Applied Mathematics Letters

www.elsevier.nl/locate/aml

The Stability of Nematic Liquid Crystals Under Crossed Electric and Magnetic Fields

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(Received February 1999; accepted August 1999)

Abstract—The alignment of the director in nematic liquid crystals is known to be influenced by electric and/or magnetic fields. The stability of known traveling wave solutions is examined for a best fit cubic equation which approximates the nonlinear sinusoidal terms in the dynamic equation for the director when it is subjected to crossed electric and magnetic fields.

Keywords—Nematic liquid crystals, Traveling waves, Stability.

1. INTRODUCTION

Nematic liquid crystals are anisotropic fluids which generally consist of elongated molecules whose average long molecular axes adopt a common direction in space described by the unit vector $\mathbf{n}$, called the director. Electric and magnetic fields or crossed combinations of both are known to influence the orientation of the director. Details of these effects and other properties of liquid crystals can be found in [1].

This letter considers the time dependent traveling wave solutions to the dynamic equation for the director $\mathbf{n}$ in a sample of nematic which is infinite in the z-direction and has crossed electric and magnetic fields applied in the xy-plane. As usual, the alignment of the director is assumed uniform in the xy-plane, the only change in orientation occurring along the z-axis. Similar problems involving only magnetic fields have been examined by de Gennes [2] and Helfrich [3] and results on related work have been reviewed by Lam [4]. In particular, we set

$$\mathbf{n} = (\cos\phi(z,t), \sin\phi(z,t), 0), \tag{1.1}$$

PII: S0893-9659(99)00203-7

where ϕ is the angle $\mathbf{n}$ makes with the x-axis. The electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ are introduced as

$$\mathbf{E} = E(\cos\beta, \sin\beta, 0), \tag{1.2}$$

$$\mathbf{H} = H(1, 0, 0), \tag{1.3}$$

where E and H are the magnitudes of the fields and β is the angle between them with $0 \leq \beta \leq \pi/2$. It was shown in detail in [5] that the continuum equations for $\mathbf{n}$ (for example, see [6]) reduce to

$$\gamma_1 \phi_t = K_2 \phi_{zz} - \frac{1}{2}\,\chi_a H^2 \sin(2\phi) - \frac{1}{2}\,\epsilon_a \epsilon_0 E^2 \sin 2(\phi - \beta), \tag{1.4}$$

where γ_1 is the twist viscosity coefficient and $K_2 > 0$ is the bulk elastic twist constant. The diamagnetic anisotropy χ_a and the dielectric anisotropy ϵ_a are assumed to be positive while the (positive) permittivity of free space is denoted by ϵ_0.

Faetti *et al.* [7] have examined equation (1.4) when the term in K_2 is omitted and the crossed fields are alternately switched on and off. The elastic term and both field terms are included below. Equation (1.4) can be written as [5]

$$\eta \phi_t = \xi^2 \phi_{zz} - \frac{1}{2}\,\sin(2\phi - q), \tag{1.5}$$

where

$$\eta = \gamma_1 \left(\epsilon_a^2 \epsilon_0^2 E^4 + \chi_a^2 H^4 + 2\epsilon_a \epsilon_0 \chi_a E^2 H^2 \cos(2\beta)\right)^{-1/2}, \tag{1.6}$$

$$\xi = \sqrt{K_2}\left(\epsilon_a^2 \epsilon_0^2 E^4 + \chi_a^2 H^4 + 2\epsilon_a \epsilon_0 \chi_a E^2 H^2 \cos(2\beta)\right)^{-1/4}, \tag{1.7}$$

$$q = \tan^{-1}\left(\frac{\epsilon_a \epsilon_0 E^2 \sin(2\beta)}{\chi_a H^2 + \epsilon_a \epsilon_0 E^2 \cos(2\beta)}\right). \tag{1.8}$$

The parameter q is the main control parameter for the problem since it not only characterizes the various possible solutions, but also, like η and ξ, links the contributions of the electric and magnetic fields with their crossed angle. Although the explicit q dependence in (1.5) can obviously be suppressed by introducing $\hat{\phi} = 2\phi - q$, we choose to retain q explicitly so that the interpretation of the results can be put into a clearer physical context.

Equation (1.5) has been investigated by the present authors using a cubic approximation in ϕ and results on exact traveling wave solutions and their stability have been reported in references [5,8], respectively. The approximation in Section 2 is a more accurate representation of (1.5) than that introduced initially by the authors and also has the added advantage of removing restrictions on the control parameter q which were necessary for ensuring traveling wave stability as discussed in [8]. The stability of traveling waves is discussed in Section 3.

2. BEST FIT CUBIC APPROXIMATION

We consider equation (1.5) with $|q| < \pi/2$ and rescale by setting $T = t/\eta$, $Z = z/\xi$ to give

$$\phi_T = \phi_{ZZ} - \frac{1}{2}\,\sin(2\phi - q). \tag{2.1}$$

In [5,8], the sine term in (2.1) was approximated via a Taylor expansion around $\phi = \pi/2$ and the roots of the approximating cubic equation satisfied

$$\phi_1 > 0, \qquad \phi_3 < \phi_2 < \phi_1. \tag{2.2}$$

These roots worked well for small q but could have a spurious dependence on q since they diverge as $q \to \pi/2$. Equation (2.2) motivates an alternative choice for the roots to equation (2.1) in an attempt to have a more accurate approximation to the sine term valid for all $|q| < \pi/2$, namely,

$$\phi_1 = \frac{q}{2}, \qquad \phi_2 = \frac{q}{2} - \frac{\pi}{2}, \qquad \phi_3 = \frac{q}{2} - \pi. \tag{2.3}$$

Defining

$$f(\phi) = \frac{1}{2}\sin(2\phi - q), \tag{2.4}$$

$$g(\phi) = A(\phi - \phi_1)(\phi - \phi_2)(\phi - \phi_3), \tag{2.5}$$

we see that for any constant A, f, and g then have the advantage of sharing the same roots, given by equation (2.3). In all cases, $\phi_3 < \phi < \phi_1$ for the traveling wave solutions which are considered. Therefore, to find the best fitting constant, we minimize the integral

$$I_A = \int_{q/2-\pi}^{q/2} [f(\phi) - g(\phi)]^2 \, d\phi \tag{2.6}$$

overall possible constants A. A simple computation shows

$$I_A = \frac{\pi^7}{840} A^2 - \frac{3}{4}\pi A + \frac{\pi}{8}, \tag{2.7}$$

and therefore, I_A is independent of q (as expected) and achieves its minimum when

$$A = 315\pi^{-6}. \tag{2.8}$$

The approximating equation to (2.1) is now

$$\phi_T = \phi_{ZZ} - g(\phi), \tag{2.9}$$

where $g(\phi)$ is the best fit cubic given by (2.5) and (2.8).

3. STABILITY OF TRAVELING WAVE SOLUTIONS

Traveling wave solutions to (2.9) were discussed in [5] by first introducing the variable

$$\tau = Z - cT + Z_0, \tag{3.1}$$

where $c = c(q) > 0$ as specified below for each solution and Z_0 is an arbitrary constant. Equation (2.9) is then

$$\phi_{\tau\tau} + c\phi_\tau = g(\phi). \tag{3.2}$$

Following [8; 9, p. 158], equation (3.2) can be written in a moving coordinate frame by changing the variables to (with the constant Z_0 set to zero)

$$T = T, \qquad \tau = Z - cT. \tag{3.3}$$

Equation (3.2) is then

$$\phi_T = \phi_{\tau\tau} + c\phi_\tau - g(\phi), \tag{3.4}$$

where in general $\phi = \phi(\tau, T)$. When $\hat{\phi}(\tau)$ is a wavefront solution to (3.2), we can investigate its stability properties by considering solutions u to (3.4) of the form

$$u(\tau, T) = \hat{\phi}(\tau) + v(\tau, T), \tag{3.5}$$

where v is a small perturbation which depends on both τ and time T. The substitution of (3.5) into (3.4) furnishes the linearized equation for v, namely,

$$v_T = v_{\tau\tau} + cv_\tau - \frac{\partial g}{\partial \hat{\phi}} v. \tag{3.6}$$

For the simplest type of stability, we employ the techniques used in [9] and examine solutions to (3.6) where

$$v(\tau, T) = 0, \qquad \text{for } |\tau| \geq L \tag{3.7}$$

for some constant $L > 0$; this means that the perturbations can be considered as vanishing outside some finite interval in the moving coordinate frame. We now seek solutions of the form

$$v(\tau, T) = v_0(\tau)e^{-\lambda T}. \tag{3.8}$$

Inserting (3.8) into (3.6) and linearizing in v results in

$$v_0'' + cv_0' + \left[\lambda - \frac{\partial g}{\partial \hat{\phi}}\right] v_0 = 0, \tag{3.9}$$

where $'$ denotes $\frac{d}{d\tau}$ and $v_0(\pm L) = 0$. Introducing the transformation

$$v_0 = w(\tau)e^{-c\tau/2} \tag{3.10}$$

reduces (3.9) to the regular Sturm-Liouville problem

$$w'' + [\lambda - Q(\tau, q)]w = 0, \qquad \text{with } w(\pm L) = 0, \tag{3.11}$$

where the values of λ are the eigenvalues and

$$Q(\tau, q) = \frac{\partial g}{\partial \hat{\phi}} + \frac{c^2}{4}. \tag{3.12}$$

It is well known (for example, see [10, p. 374]) that $\lambda > 0$, whenever Q is continuous for all $\tau \in [-L, L]$ and

$$Q(\tau, q) > 0, \qquad \text{for all } \tau \in [-L, L]. \tag{3.13}$$

When (3.13) holds, λ can only take positive values and it is then clear from (3.8) and (3.10) that

$$v(\tau, T) \to 0, \qquad \text{as } T \to \infty, \tag{3.14}$$

and therefore, the wavefront solution $\hat{\phi}$ is linearly stable. It is observed that Q is a quadratic in $\hat{\phi}$ and that its roots are given by, after some simple calculations,

$$Q_\pm = \frac{1}{2}(q - \pi) \pm \frac{\pi}{2\sqrt{3}} \sqrt{1 - A^{-1}\pi^{-2}c^2}. \tag{3.15}$$

It follows that $Q(\tau, q) > 0$ for all τ (and therefore, for all fixed $L > 0$), whenever

$$c^2 > A\pi^2 = 315\pi^{-4}, \tag{3.16}$$

where A is given by (2.8). Hence, whenever $c(q)$ satisfies (3.16), we can deduce that (3.13) is valid and conclude that (3.14) must hold, indicating that the perturbations v must decay to zero as $T \to \infty$. Equation (3.16) provides a criterion involving the wavespeed c to ensure linear stability for wavefront solutions $\hat{\phi}$. We now consider the three types of traveling waves discussed in [5,8], which are stated explicitly for convenience.

Type A traveling waves are of the form

$$\hat{\phi} = (\phi_1 - \phi_2)\left\{1 + \exp\left[-\frac{1}{\sqrt{2}}(\phi_1 - \phi_2)\tau\right]\right\}^{-1} + \phi_2, \tag{3.17}$$

and solve equation (3.2). Here, $\hat{\phi}$ travels from ϕ_1 to ϕ_2 as $\tau \to \infty$ and

$$c = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2 - 2\phi_3) = \frac{3\pi}{2\sqrt{2}}. \tag{3.18}$$

Hence, condition (3.16) holds, and therefore, perturbations decay to zero. This is a major improvement to the result in [8] where a restriction on the possible values for q is necessary for stability.

For type B traveling waves, we have

$$\hat{\phi} = (\phi_2 - \phi_3)\left\{1 + \exp\left[-\frac{1}{\sqrt{2}}(\phi_2 - \phi_3)\tau\right]\right\}^{-1} + \phi_3, \tag{3.19}$$

where $\hat{\phi}$ travels from ϕ_3 to ϕ_2 as $\tau \to \infty$ and

$$c = -\frac{1}{\sqrt{2}}(\phi_2 + \phi_3 - 2\phi_1) = \frac{3\pi}{2\sqrt{2}}. \tag{3.20}$$

Again, since (3.16) holds, these traveling waves are linearly stable.

For type C, there is a static (T independent) solution of the form

$$\hat{\phi} = (\phi_1 - \phi_3)\left\{1 + \exp\left[-\frac{1}{\sqrt{2}}(\phi_1 - \phi_3)\tau\right]\right\}^{-1} + \phi_3, \tag{3.21}$$

with $\hat{\phi}$ traveling from ϕ_1 to ϕ_3 and

$$c = \frac{1}{\sqrt{2}}(\phi_1 + \phi_3 - 2\phi_2) = 0. \tag{3.22}$$

It is simple to verify that for all $|q| < \pi/2$, the function $Q(\tau, q)$ must become negative at some finite value of τ because $\phi_1 > 0$ and $\phi_3 < 0$: for example, take $\tau = 0$ in (3.21) and insert $\hat{\phi}$ into (3.12) with $c = 0$ to find $Q(0, q) = -A\pi^2/4$. In this case, the positivity of Q cannot hold for all τ and so stability in general cannot be guaranteed by the methods presented here for the static type C solutions.

4. SUMMARY

A best fit cubic approximation for the sine term in (1.5), which has roots ϕ_1, ϕ_2, ϕ_3 given by (2.3), has been exploited to derive stability results for known traveling wave solutions in nematics as discussed in Section 3. These solutions for the orientation of the director arise when crossed static electric and magnetic fields are applied across a sample of nematic. Equation (3.16) provides a simple stability criterion involving the wavespeed c. The type A solution in (3.17) travels from ϕ_1 to ϕ_2 and is linearly stable to perturbations v which obey (3.7). Similarly, the type B solution in (3.19) travels from ϕ_3 to ϕ_2 and is linearly stable. The type C solution in (3.21) is a static domain wall and its stability cannot in general be guaranteed since $Q(\tau, q)$ can take negative values, and therefore, the eigenvalues for the perturbation equation (3.11) may not necessarily be positive.

REFERENCES

1. P.G. de Gennes and J. Prost, *The Physics of Liquid Crystals*, Clarendon Press, Oxford, (1993).
2. P.G. de Gennes, Mouvements de parois dans un nématique sous champ tournant, *J. Phys. (Paris)* **32**, 789, (1971).
3. W. Helfrich, Alignment-inversion walls in nematic liquid crystals in the presence of a magnetic field, *Phys. Rev. Letters* **21**, 1518, (1968).
4. L. Lam, Solitons and field induced solitons in liquid crystals, In *Solitons in Liquid Crystals*, (Edited by L. Lam and J. Prost), Springer-Verlag, New York, (1992).
5. I.W. Stewart and T.R. Faulkner, Director orientation in nematic liquid crystals using crossed electric and magnetic fields, *J. Phys. A: Math. Gen.* **28**, 5643, (1995).
6. F.M. Leslie, Some topics in equilibrium theory of liquid crystals, In *Theory and Applications of Liquid Crystals*, (Edited by J.L. Ericksen and D. Kinderlehrer), Springer-Verlag, New York, (1987).
7. S. Faetti, L. Gammaitoni, M. Martinelli and P.A. Rolla, Director reorientation in nematics studied by microwave dielectrometry, *Liq. Cryst.* **9**, 133, (1991).
8. I.W. Stewart and T.R. Faulkner, The stability of travelling waves induced by crossed electric and magnetic fields in nematic liquid crystals, *Cont. Mech. Thermodyn.* **9**, 191, (1997).
9. J.D. Logan, *An Introduction to Nonlinear Partial Differential Equations*, Wiley, New York, (1994).
10. J.L. Troutman, *Boundary Value Problems of Applied Mathematics*, PWS Publishing, Boston, MA, (1994).